Number theory and Cryptography

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Definition (Primitive Root)

A *primitive root* modulo an integer n is an element of $(\mathbf{Z}/n\mathbf{Z})^*$ of order $\varphi(n)$.

The Ring of Integers Modulo n

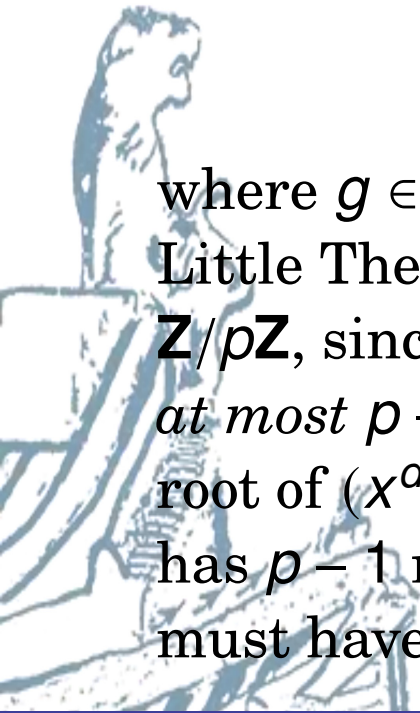
Proposition

Let p be a prime number and let d be a divisor of $p-1$. Then $f = x^d - 1 \in (\mathbf{Z}/p\mathbf{Z})[x]$ has exactly d roots in $\mathbf{Z}/p\mathbf{Z}$.

Proof.

Let $e = (p-1)/d$. We have

$$\begin{aligned} x^{p-1} - 1 &= (x^d)^e - 1 \\ &= (x^d - 1)((x^d)^{e-1} + (x^d)^{e-2} + \cdots + 1) \\ &= (x^d - 1)g(x), \end{aligned}$$



where $g \in (\mathbf{Z}/p\mathbf{Z})[x]$ and $\deg(g) = de - d = p - 1 - d$. Fetmat's Little Theorem implies that $x^{p-1} - 1$ has exactly $p-1$ roots in $\mathbf{Z}/p\mathbf{Z}$, since every nonzero element of $\mathbf{Z}/p\mathbf{Z}$ is a root! Since g has *at most* $p-1-d$ roots and $x^d - 1$ has at most d roots. Since a root of $(x^d - 1)g(x)$ is a root of either $x^d - 1$ or $g(x)$ and $x^{p-1} - 1$ has $p-1$ roots, g must have exactly $p-1-d$ roots and $x^d - 1$ must have exactly d roots, as claimed. □

Lemma

Suppose $a, b \in (\mathbf{Z}/n\mathbf{Z})^*$ *have orders* r *and* s*, respectively, and that* $\gcd(r, s) = 1$*. Then* ab *has order* rs*.*

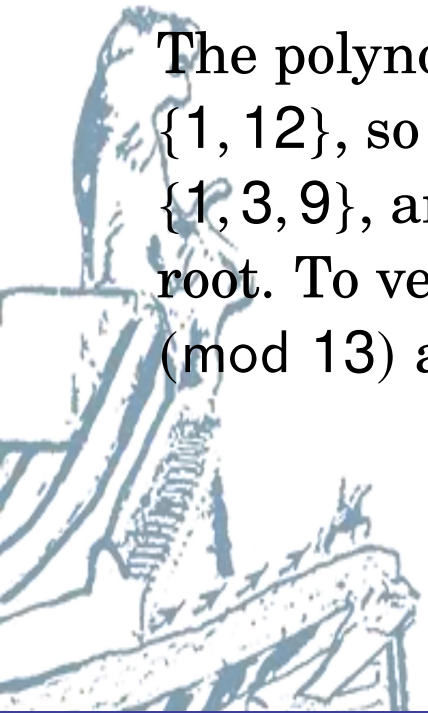
Theorem (Primitive Roots)

There is a primitive root modulo any prime p. In particular, the group $(\mathbf{Z}/p\mathbf{Z})^*$ is cyclic.

Example

We illustrate the proof of the Primitive Roots Theorem when $p = 13$. We have

$$p - 1 = 12 = 2^2 \cdot 3.$$



The polynomial $x^4 - 1$ has roots $\{1, 5, 8, 12\}$ and $x^2 - 1$ has roots $\{1, 12\}$, so we may take $a_1 = 5$. The polynomial $x^3 - 1$ has roots $\{1, 3, 9\}$, and we set $a_2 = 3$. Then $a = 5 \cdot 3 = 15 \equiv 2$ is a primitive root. To verify this, note that the successive powers of 2 (mod 13) are

$$2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1.$$

Example

Primitive Roots Theorem is false if, for example, p is replaced by a power of 2 bigger than 4. For example, the four elements of $(\mathbf{Z}/8\mathbf{Z})^*$ each have order dividing 2, but $\varphi(8) = 4$.

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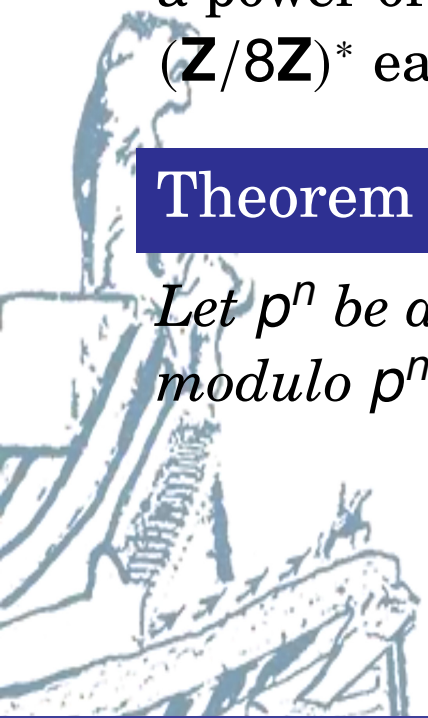
Theorem (Primitive Roots mod p^n)

Let p^n be a power of an odd prime. Then there is a primitive root modulo p^n.



Proposition (Number of Primitive Roots)

If there is a primitive root modulo n, then there are exactly $\varphi(\varphi(n))$ primitive roots modulo n.

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Proof.

The primitive roots modulo n are the generators of $(\mathbf{Z}/n\mathbf{Z})^*$, which by assumption is cyclic of order $\varphi(n)$. Thus they are in bijection with the generators of any cyclic group of order $\varphi(n)$. In particular, the number of primitive roots modulo n is the same as the number of elements of $\mathbf{Z}/\varphi(n)\mathbf{Z}$ with additive order $\varphi(n)$. An element of $\mathbf{Z}/\varphi(n)\mathbf{Z}$ has additive order $\varphi(n)$ if and only if it is coprime to $\varphi(n)$. There are $\varphi(\varphi(n))$ such elements, as claimed. □

Example

For example, there are $\varphi(\varphi(17)) = \varphi(16) = 2^4 - 2^3 = 8$ primitive roots mod 17, namely $3, 5, 6, 7, 10, 11, 12, 14$. The $\varphi(\varphi(9)) = \varphi(6) = 2$ primitive roots modulo 9 are 2 and 5. There are no primitive roots modulo 8, even though $\varphi(\varphi(8)) = \varphi(4) = 2 > 0$.