Number theory and Cryptography

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Theorem (Chinese Remainder Theorem)

Let $a, b \in \mathbf{Z}$ *and* $n, m \in \mathbf{N}$ *such that* $\gcd(n, m) = 1$. *Then there exists* $x \in \mathbf{Z}$ *such that*

$$x \equiv a \pmod{m},$$
$$x \equiv b \pmod{n}.$$

Moreover x *is unique modulo* mn.

Proof.

If we can solve for t in the equation

$$a + tm \equiv b \pmod{n},$$

then $x = a + tm$ will satisfy both congruences. To see that we can solve, subtract a from both sides and use the Proposition of Unit together with our assumption that $\gcd(n, m) = 1$ to see that there is a solution.
For uniqueness, suppose that x and y solve both congruences. Then $z = x - y$ satisfies $z \equiv 0 \pmod{m}$ and $z \equiv 0 \pmod{n}$, so $m \mid z$ and $n \mid z$. Since $\gcd(n, m) = 1$, it follows that $nm \mid z$, so $x \equiv y \pmod{nm}$. □

Lemma

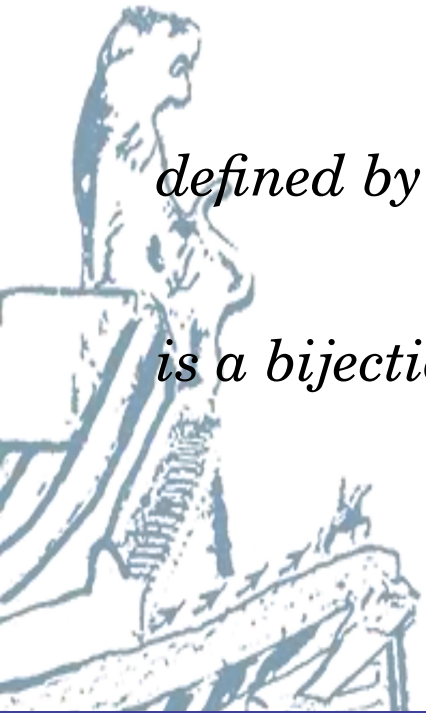
Suppose that $m, n \in \mathbf{N}$ *and* $\gcd(m, n) = 1$. *Then the map*

$$\psi : (\mathbf{Z}/mn\mathbf{Z})^* \to (\mathbf{Z}/m\mathbf{Z})^* \times (\mathbf{Z}/n\mathbf{Z})^*. \tag{1}$$

defined by

$$\psi(c) = (c \bmod m,\ c \bmod n)$$

is a bijection.



Definition (Multiplicative Function)

A function $f : \mathbf{N} \to \mathbf{C}$ is *multiplicative* if, whenever $m, n \in \mathbf{N}$ and $\gcd(m, n) = 1$, we have

$$f(mn) = f(m) \cdot f(n).$$

Proposition (Multiplicativity of φ)

The function φ is multiplicative.

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Proof.

The map ψ of Lemma 2 is a bijection, so the set on the left in (1) has the same size as the product set on the right in (1). Thus

$$\varphi(mn) = \varphi(m) \cdot \varphi(n).$$

□

Proposition (Extended Euclidean Representation)

Suppose $a, b \in \mathbf{Z}$ *and let* $g = \gcd(a, b)$. *Then there exists* $x, y \in \mathbf{Z}$ *such that*

$$ax + by = g.$$

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Proof.

Let $g = \gcd(a, b)$. Then $\gcd(a/g, b/g) = 1$, so by the Solvability Proposition, the equation

$$\frac{a}{g} \cdot x \equiv 1 \left(\mathrm{mod}\ \frac{b}{g}\right) \tag{2}$$

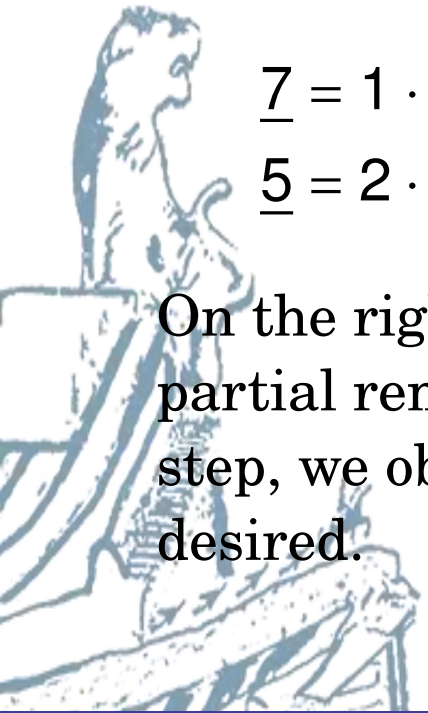
has a solution $x \in \mathbf{Z}$. Multiplying (2) through by g yields $ax \equiv g \pmod{b}$, so there exists y such that $b \cdot (-y) = ax - g$. Then $ax + by = g$, as required.

Example

Suppose $a = 5$ and $b = 7$. Here we underline certain numbers, because it clarifies the subsequent back substitution we will use to find x and y.

$$\underline{7} = 1 \cdot \underline{5} + \underline{2} \qquad \text{so } \underline{2} = \underline{7} - \underline{5}$$

$$\underline{5} = 2 \cdot \underline{2} + \underline{1} \qquad \text{so } \underline{1} = \underline{5} - 2 \cdot \underline{2} = \underline{5} - 2(\underline{7} - \underline{5}) = 3 \cdot \underline{5} - 2 \cdot \underline{7}$$



On the right, we have back-substituted in order to write each partial remainder as a linear combination of a and b. In the last step, we obtain $\gcd(a, b)$ as a linear combination of a and b, as desired.

Example

That example was not too complicated, so we try another one. Let $a = 130$ and $b = 61$. We have

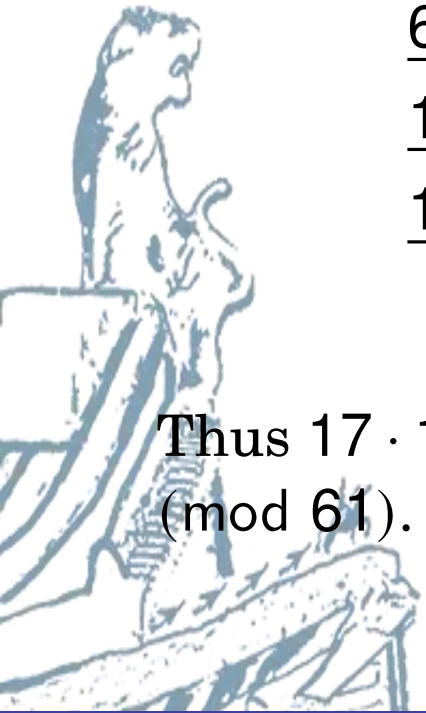
$$\begin{aligned} \underline{130} &= 2 \cdot \underline{61} + \underline{8} & \underline{8} &= \underline{130} - 2 \cdot \underline{61} \\ \underline{61} &= 7 \cdot \underline{8} + \underline{5} & \underline{5} &= -7 \cdot \underline{130} + 15 \cdot \underline{61} \\ \underline{8} &= 1 \cdot \underline{5} + \underline{3} & \underline{3} &= 8 \cdot \underline{130} - 17 \cdot \underline{61} \\ \underline{5} &= 1 \cdot \underline{3} + \underline{2} & \underline{2} &= -15 \cdot \underline{130} + 32 \cdot \underline{61} \\ \underline{3} &= 1 \cdot \underline{2} + \underline{1} & \underline{1} &= 23 \cdot \underline{130} - 49 \cdot \underline{61} \end{aligned}$$

Thus $x = 23$ and $y = -49$ is a solution to $130x + 61y = 1$.

Example

Solve $17x \equiv 1 \pmod{61}$. First, we use the Euclid Algorithm to find x, y such that $17x + 61y = 1$:

$$\begin{aligned} \underline{61} &= 3 \cdot \underline{17} + \underline{10} \\ \underline{17} &= 1 \cdot \underline{10} + \underline{7} \\ \underline{10} &= 1 \cdot \underline{7} + \underline{3} \\ \underline{3} &= 2 \cdot \underline{3} + \underline{1} \end{aligned} \qquad \begin{aligned} \underline{10} &= \underline{61} - 3 \cdot \underline{17} \\ \underline{7} &= -\underline{61} + 4 \cdot \underline{17} \\ \underline{3} &= 2 \cdot \underline{61} - 7 \cdot \underline{17} \\ \underline{1} &= -5 \cdot \underline{61} + 18 \cdot \underline{17} \end{aligned}$$



Thus $17 \cdot 18 + 61 \cdot (-5) = 1$ so $x = 18$ is a solution to $17x \equiv 1 \pmod{61}$.

Theorem (Pseudoprimality)

An integer $p > 1$ *is prime if and only if for* every $a \not\equiv 0 \pmod{p}$,

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Proof.

If p is prime, then the statement follows from Fermat's Little Theorem. If p is composite, then there is a divisor a of p with $2 \leq a < p$. If $a^{p-1} \equiv 1 \pmod{p}$, then $p \mid a^{p-1} - 1$. Since $a \mid p$, we have $a \mid a^{p-1} - 1$, hence there exists an integer k such that $ak = a^{p-1} - 1$. Subtracting, we see that $a^{p-1} - ak = 1$, so $a(a^{p-2} - k) = 1$. This implies that $a \mid 1$, which is a contradiction since $a \geq 2$. □

Example

Is $p = 323$ prime? We compute $2^{322} \pmod{323}$. Making a table as above, we have

i	m	ε_i	$2^{2^i} \bmod 323$
0	322	0	2
1	161	1	4
2	80	0	16
3	40	0	256
4	20	0	290
5	10	0	120
6	5	1	188
7	2	0	137
8	1	1	35

Example

Thus

$$2^{322} \equiv 4 \cdot 188 \cdot 35 \equiv 157 \pmod{323},$$

so 323 is not prime, though this computation gives no information about how 323 factors as a product of primes. In fact, one finds that $323 = 17 \cdot 19$.